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A Variational Principle for the Stream Function–Vorticity Formulation of the Navier–Stokes Equations Incorporating No-Slip Conditions

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A variational principle is derived for the stream function–vorticity formulation of the Navier–Stokes equations which has the no-slip boundary conditions as natural boundary conditions. The usual difficulty of determining a finite difference approximation for the boundary vorticity at a rigid wall can be avoided by a finite element approach. Piecewise bilinear trial functions on rectangles are shown to give a generalization of a formula due to Woods. The same choice also gives a second order accurate Arakawa scheme for the Jacobian of the stream function and vorticity.

1. INTRODUCTION

There is an extensive and rapidly growing literature on the application of finite element methods to the numerical solution of fluid flow problems. In many cases, such as potential flow, compressible inviscid flow, and slow viscous flow there are well-established minimum principles or complementary variational principles which serve as a sound starting point for the analysis. In most problems there is no known minimum principle so it is usual to employ the Galerkin method with the expansion functions as piecewise low order polynomials with compact support on small regions. Recently Usher and Craik [1] have renewed interest in a variational principle for the full Navier–Stokes equations due to Bateman [2], which uses the original velocity and pressure fields plus pseudovelocity and pseudopressure fields. The complete set of Euler–Lagrange equations consists of the Navier–Stokes equations and their adjoints. Finlayson [3] had earlier severely criticized the use of such variational principles on the grounds of their having no physical interpretation and producing no new numerical solution procedures. Usher and Craik, in spite of this comment, have used the Bateman principle usefully in discussing nonlinear wave interactions.

In this paper a variational principle in the spirit of Bateman is given for the stream function–vorticity formulation of the Navier–Stokes equations which has only the values of the stream function at a solid boundary and the vorticity at a free boundary as essential boundary conditions; the *no-slip* conditions on the normal derivative of the stream function at a solid boundary are *natural* boundary conditions. The full

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power of the finite element method is then used to rederive the Arakawa second order scheme for the convective terms in the vorticity equation. This scheme emerges by simply assuming that the stream function and vorticity are piecewise bilinear on rectangles as has been shown recently by Jespersen [4]. From this same assumption some interesting new approximations for the boundary vorticity are obtained. Previous authors have approximated boundary vorticities solely in terms of quantities along the normal to the boundary. Here it is shown that it is more natural to include tangential contributions as well.

2. The Variational Principle

It is supposed that incompressible viscous fluid of constant density is flowing steadily in a region D of the (x, y) plane with boundary ∂D. If the governing equations are made nondimensional with respect to characteristic length and velocity scales the equations for the dimensionless stream function Ψ^* and vorticity Ω^* which apply within D are

$$\nabla^2\Omega^* - R_e\left(\frac{\partial\Psi^*}{\partial y}\frac{\partial\Omega^*}{\partial x} - \frac{\partial\Psi^*}{\partial x}\frac{\partial\Omega^*}{\partial y}\right) = 0, \tag{2.1}$$

$$\nabla^2\Psi^* + \Omega^* = 0, \tag{2.2}$$

where R_e is the Reynolds number. In a typical problem the boundary ∂D may be supposed to consist of sections ∂D_0 which are stationary and solid, sections ∂D_f which are free, and sections ∂D_m which are moving and solid. Boundary conditions are

$$\Psi^* = \Psi_0, \qquad \partial\Psi^*/\partial n = 0 \text{ on } \partial D_0; \tag{2.3}$$

$$\Psi^* = \Psi_f, \qquad \Omega = \Omega_f \text{ on } \partial D_f; \tag{2.4}$$

$$\Psi^* = \Psi_m, \qquad \partial\Psi^*/\partial n = U \text{ on } \partial D_m, \tag{2.5}$$

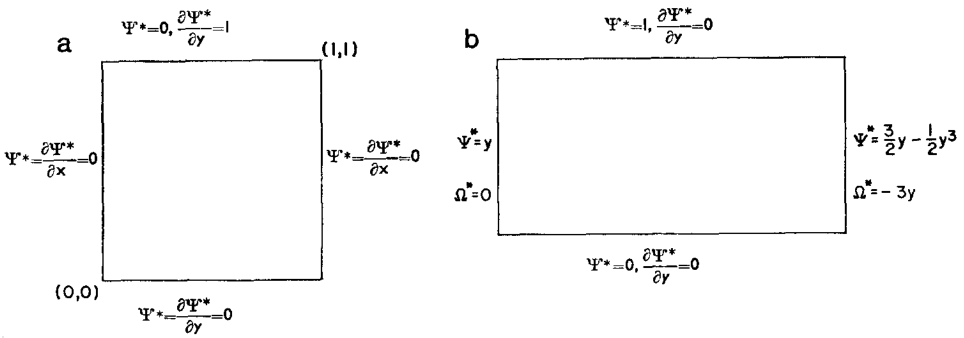

FIG. 1. Sample problems. (a) Driven cavity flow; three sides form ∂D_0, top forms ∂D_m. (b) Entrance flow; top and bottom form ∂D_0, sides form ∂D_f.

where $\partial/\partial n$ denotes differentiation along the outward normal. Ψ_0 may be assumed to be constant on ∂D_0, Ψ_m and U assumed to be constant on ∂D_m, and Ψ_f and Ω_f assumed to be prescribed functions of x and y. As examples, the boundary conditions for a driven cavity flow and an entrance length flow in a duct are shown in Figs. 1a and 1b. For simplicity of presentation, attention will be restricted initially to problems where there are no moving boundaries.

It will now be established that there is a functional which has a stationary point at the solution of the problem given by Eqs. (2.1)–(2.4). Consider the functional

$$J(\psi_1, \psi_2, \omega_1, \omega_2) = \int_D L\, da \tag{2.6}$$

of the two pseudostream functions ψ_1 and ψ_2 and the pseudovorticities ω_1 and ω_2, over the region D with area element da, where

$$\begin{aligned} L(\psi_1, \psi_2, \omega_1, \omega_2) &= \frac{\partial\psi_1}{\partial x}\frac{\partial\omega_1}{\partial x} + \frac{\partial\psi_1}{\partial y}\frac{\partial\omega_1}{\partial y} - \frac{R}{2}\left(\frac{\partial\psi_1}{\partial y}\frac{\partial\omega_1}{\partial x} - \frac{\partial\psi_1}{\partial x}\frac{\partial\omega_1}{\partial y}\right)\psi_2 - \frac{\omega_1^2}{2} \\ &\quad - \frac{\partial\psi_2}{\partial x}\frac{\partial\omega_2}{\partial x} - \frac{\partial\psi_2}{\partial y}\frac{\partial\omega_2}{\partial y} + \frac{R}{2}\left(\frac{\partial\psi_2}{\partial y}\frac{\partial\omega_2}{\partial x} - \frac{\partial\psi_2}{\partial x}\frac{\partial\omega_2}{\partial y}\right)\psi_1 + \frac{\omega_2^2}{2}. \end{aligned}$$

If ψ_1, ψ_2, ω_1, and ω_2, belonging to the class of functions having piecewise continuous second derivatives and bounded Sobolev norms such that

$$\int_D (\operatorname{grad} \psi_i)^2\, da \leqslant M_i^2, \qquad i = 1, 2 \tag{2.7}$$

are subjected to independent variations, then the first variation in J,

$$\begin{aligned} \delta J = \int_D \Big\{ &\delta\left(\frac{\partial\psi_1}{\partial x}\right)\left(\frac{\partial\omega_1}{\partial x} + \frac{R}{2}\frac{\partial\omega_1}{\partial y}\psi_2\right) + \delta\left(\frac{\partial\psi_1}{\partial y}\right)\left(\frac{\partial\omega_1}{\partial y} - \frac{R}{2}\frac{\partial\omega_1}{\partial x}\psi_2\right) \\ &+ \delta\left(\frac{\partial\omega_1}{\partial x}\right)\left(\frac{\partial\psi_1}{\partial x} - \frac{R}{2}\frac{\partial\psi_1}{\partial y}\psi_2\right) + \delta\left(\frac{\partial\omega_1}{\partial y}\right)\left(\frac{\partial\psi_1}{\partial y} + \frac{R}{2}\frac{\partial\psi_1}{\partial x}\psi_2\right) \\ &+ \frac{R}{2}\delta\psi_1\left(\frac{\partial\psi_2}{\partial y}\frac{\partial\omega_2}{\partial x} - \frac{\partial\psi_2}{\partial x}\frac{\partial\omega_2}{\partial y}\right) - \delta\omega_1\,\omega_1\Big\}\, da \\ &- \text{corresponding terms in } \delta\psi_2, \delta\omega_2, \text{ and derivatives,} \end{aligned}$$

can be written

$$\begin{aligned} \delta J = \int_D \Big\{ &\delta\psi_1\left[-\frac{\partial^2\omega_1}{\partial x^2} - \frac{\partial^2\omega_1}{\partial y^2} + \frac{R}{2}\left(\frac{\partial\omega_1}{\partial x}\frac{\partial\psi_2}{\partial y} - \frac{\partial\omega_1}{\partial y}\frac{\partial\psi_2}{\partial x} + \frac{\partial\omega_2}{\partial x}\frac{\partial\psi_2}{\partial y} - \frac{\partial\omega_2}{\partial y}\frac{\partial\psi_2}{\partial x}\right)\right] \\ &+ \delta\omega_1\left[-\frac{\partial^2\psi_1}{\partial x^2} - \frac{\partial^2\psi_1}{\partial y^2} + \frac{R}{2}\left(\frac{\partial\psi_1}{\partial y}\frac{\partial\omega_2}{\partial x} - \frac{\partial\psi_2}{\partial y}\frac{\partial\omega_1}{\partial x}\right) - \omega_1\right]\Big\}\, da \\ &+ \int_{\partial D} \Big\{\delta\psi_1\left(\frac{\partial\omega_1}{\partial n} + \frac{R}{2}\frac{\partial\omega_1}{\partial s}\psi_2\right) + \delta\omega_1\left(\frac{\partial\psi_1}{\partial n} - \frac{R}{2}\frac{\partial\psi_1}{\partial s}\psi_2\right)\Big\}\, ds \\ &- \text{corresponding terms in } \delta\psi_2, \delta\omega_2. \end{aligned}$$

The Euler–Lagrange equations are

$$\delta\psi_1 : \nabla^2\omega_1 - \frac{R}{2}\left[\frac{\partial\psi_2}{\partial y}\left(\frac{\partial\omega_1}{\partial x} + \frac{\partial\omega_2}{\partial x}\right) - \left(\frac{\partial\psi_2}{\partial x}\right)\left(\frac{\partial\omega_1}{\partial y} + \frac{\partial\omega_2}{\partial y}\right)\right] = 0, \tag{2.8}$$

$$\delta\omega_1 : \nabla^2\psi_1 + \omega_1 - \frac{R}{2}\left(\frac{\partial\psi_2}{\partial x}\frac{\partial\psi_1}{\partial y} - \frac{\partial\psi_2}{\partial y}\frac{\partial\psi_1}{\partial x}\right) = 0, \tag{2.9}$$

and the natural boundary conditions on ∂D are

$$\delta\psi_1 : \frac{\partial\omega_1}{\partial n} + \frac{R}{2}\frac{\partial\omega_1}{\partial s}\psi_2 = 0, \tag{2.10}$$

$$\delta\omega_1 : \frac{\partial\psi_1}{\partial n} - \frac{R}{2}\frac{\partial\psi_1}{\partial s}\psi_2 = 0, \tag{2.11}$$

with two further equations and two sets of natural boundary conditions with the roles of (ψ_1, ω_1) and (ψ_2, ω_2) interchanged.

The next stage of the argument is to show that if the corresponding pseudovariables satisfy the same Dirichlet conditions on the appropriate parts of the boundary ∂D then they are equal throughout D and satisfy certain natural boundary conditions, in which case, Eqs. (2.8) and (2.9) reduce to Eqs. (2.1) and (2.2), and a variational principle is established for a problem which has the *same solution* as that posed by Eqs. (2.1)–(2.4), provided the correct natural boundary conditions arise.

Suppose then that the pseudovariable trial functions satisfy the essential conditions

$$\psi_1 = \psi_2 = \Psi_0 \text{ on } \partial D_0 \quad \text{and} \quad \psi_1 = \psi_2 = \Psi_f \text{ on } \partial D_f, \tag{2.10a}$$

$$\omega_1 = \omega_2 = \Omega_f \text{ on } \partial D_f. \tag{2.11a}$$

In standard variational principles Dirichlet boundary conditions are almost invariably regarded as essential conditions to be satisfied by the trial functions, so there is no real loss of generality. Dirichlet conditions are also very easy to impose in a numerical scheme. When the pseudovariables satisfy (2.10a) and (2.11a), (2.10) does not apply anywhere on ∂D and Eq. (2.11) applies on ∂D_0. As ψ_1 is piecewise constant along ∂D_0 Eq. (2.11) simplifies to

$$\partial\psi_1/\partial n = 0,$$

the analog to the no-slip condition. Similarly it may be shown that

$$\partial\psi_2/\partial n = 0 \text{ on } \partial D_0.$$

The semidifference functions

$$\bar{\Psi} = \tfrac{1}{2}(\psi_1 - \psi_2), \qquad \bar{\Omega} = \tfrac{1}{2}(\omega_1 - \omega_2)$$

satisfy the homogeneous boundary value problem

$$\nabla^2\Psi + \bar{\Omega} - R\left(\frac{\partial \Psi}{\partial y}\frac{\partial \mathit{\Psi}}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial \mathit{\Psi}}{\partial y}\right) = 0, \tag{2.12}$$

$$\nabla^2\bar{\Omega} - R\left(\frac{\partial \Omega}{\partial y}\frac{\partial \Psi}{\partial x} - \frac{\partial \Omega}{\partial x}\frac{\partial \Psi}{\partial y}\right) = 0, \tag{2.13}$$

$$\Psi = \frac{\partial \Psi}{\partial n} = 0 \text{ on } \partial D_0, \qquad \Psi = \bar{\Omega} = 0 \text{ on } \partial D_f,$$

where $\mathit{\Psi} = \frac{1}{2}(\psi_1 + \psi_2)$, $\Omega = \frac{1}{2}(\omega_1 + \omega_2)$.

The operator ∇^2 applied to (2.12) yields

$$\nabla^4\Psi = -\nabla^2\bar{\Omega} + R\nabla^2\left(\frac{\partial \Psi}{\partial y}\frac{\partial \mathit{\Psi}}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial \mathit{\Psi}}{\partial y}\right)$$

$$= -R\left(\frac{\partial \Omega}{\partial y}\frac{\partial \Psi}{\partial x} - \frac{\partial \Omega}{\partial x}\frac{\partial \Psi}{\partial y}\right) + R\nabla^2\left(\frac{\partial \Psi}{\partial y}\frac{\partial \mathit{\Psi}}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial \mathit{\Psi}}{\partial y}\right)$$

by (2.13). Consequently

$$\int_D \Psi\nabla^4\Psi da = -\frac{R}{2}\int_D \frac{\partial \Omega}{\partial y}\frac{\partial \Psi^2}{\partial x} - \frac{\partial \Omega}{\partial x}\frac{\partial \Psi^2}{\partial y}\, da$$

$$+ R\int_D \Psi\nabla^2\left(\frac{\partial \mathit{\Psi}}{\partial x}\frac{\partial \Psi}{\partial y} - \frac{\partial \mathit{\Psi}}{\partial y}\frac{\partial \Psi}{\partial x}\right) da$$

which by Gauss' divergence theorem implies

$$\int_D -\frac{\partial \Psi}{\partial x_i}\frac{\partial}{\partial x_i}(\nabla^2\Psi)\, da = -R\int_D \frac{\partial \Psi}{\partial x_i}\frac{\partial}{\partial x_i}\left(\frac{\partial \Psi}{\partial y}\frac{\partial \mathit{\Psi}}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial \mathit{\Psi}}{\partial y}\right) da.$$

(The surface integral vanishes as $\Psi = 0$ on the boundary; and the volume integral is clearly also zero. The repeated suffix summation is being used.) A further application of Gauss' theorem gives

$$\int_D (\nabla^2\Psi)^2\, da - R\int_D \nabla^2\Psi\left(\frac{\partial \Psi}{\partial y}\frac{\partial \mathit{\Psi}}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial \mathit{\Psi}}{\partial y}\right) da$$

$$= \int_{\partial D} \frac{\partial \Psi}{\partial n}\nabla^2\Psi\, ds - R\int_{\partial D} \frac{\partial \Psi}{\partial n}\left(\frac{\partial \Psi}{\partial y}\frac{\partial \mathit{\Psi}}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial \mathit{\Psi}}{\partial y}\right) ds$$

$$= -\int_{\partial D} \frac{\partial \Psi}{\partial n}\bar{\Omega}\, ds \qquad \text{(from 2.12).}$$

This surface integral vanishes because either $\partial\Psi/\partial n$ or $\bar{\Omega}$ is zero at each point of the boundary.

An application of Schwarz's inequality gives

$$\int_D (\nabla^2\Psi)^2\, da = R\left|\int_D \nabla^2\Psi\left(\frac{\partial\Psi}{\partial y}\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial\Psi}{\partial y}\right) da\right|$$
$$\leqslant R\left(\int_D (\nabla^2\Psi)^2\, da\right)^{1/2}\left(\int_D \left(\frac{\partial\Psi}{\partial y}\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial\Psi}{\partial y}\right)^2 da\right)^{1/2}$$
$$\leqslant R\left(\int_D (\nabla^2\Psi)^2\, da\right)^{1/2}\left(\int_D (\text{grad}\,\Psi)^2(\text{grad}\,\Psi)^2\, da\right)^{1/2},$$

Now the minimum value of

$$\text{grad}(\nabla^2\phi - \mu\phi) = 0, \qquad \phi = 0 \text{ on } \partial D$$

among the class of L_2 integral functions is the least eigenvalue μ_1 for the problem

$$\int_D (\nabla^2\phi)^2\, da \Big/ \int_D (\text{grad}\,\phi)^2\, da, \qquad \phi = 0 \text{ on } \partial D$$

so that

$$\int_D (\text{grad}\,\Psi)^2(\text{grad}\,\Psi)^2\, da \leqslant M^2\int_D (\text{grad}\,\Psi)^2\, da \leqslant \frac{M^2}{\mu_1}\int_D (\nabla^2\Psi)^2\, da,$$

where $M^2 \leqslant M_1^2 + M_2^2$. Thus finally

$$\int_D (\nabla^2\Psi)^2\, da \leqslant \frac{MR}{\mu^{1/2}}\int (\nabla^2\Psi)^2\, da. \tag{2.14}$$

Equation (2.14) shows that if R is small there is a contradiction unless

$$\int_D (\nabla^2\Psi)^2\, da = 0;$$

i.e., unless $\Psi = 0$ and by implication that $\bar{\Omega} = 0$. In such circumstances $\psi_1 = \psi_2 = \Psi$, $\omega_1 = \omega_2 = \Omega$, and the boundary value problem given by Eqs. (2.8), (2.9), (2.10), (2.11), (2.10a), and (2.11a) becomes

$$\nabla^2\Omega - R\left(\frac{\partial\Psi}{\partial y}\frac{\partial\Omega}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial\Omega}{\partial y}\right) = 0, \tag{2.15}$$

$$\nabla^2\Psi + \Omega = 0, \tag{2.16}$$

$$\Psi = \Psi_0, \qquad \frac{\partial\Psi}{\partial n} = 0 \text{ on } \partial D_0, \tag{2.17}$$

$$\Psi = \Psi_f, \qquad \Omega = \Omega_f \text{ on } \partial D_f. \tag{2.18}$$

The solution of this problem is the same as that originally posed provided Eqs. (2.1)–(2.4) have a unique solution and R is equal to the Reynolds number R_e.

If the solid part of the boundary has a moving section ∂D_m, this problem can be treated by adding the surface integral

$$-\int_{\partial D_m} U(\omega_1 - \omega_2)\, ds$$

to the basic functional and insisting that the trial functions $\psi_1 = \psi_2 = \Psi_m$ on ∂D_m. The boundary conditions are then modified to include

$$\Psi = \Psi_m, \qquad \partial\Psi/\partial n = U \text{ on } \partial D_m.$$

3. Application of Finite Elements

The variational principle established in the previous section will now be used to derive a finite element approximation for the stream function–vorticity equations for two-dimensional flow. It will be supposed that the flow region D can be divided into small square regions and that a typical node labeled 0 is surrounded by nodes 1–8 as shown in Fig. 2 and that local coordinates X, Y are measured from 0. On the element bounded by nodes 0–3 it will be supposed that

$$\psi_m{}^e = \sum_{i=0}^{3} \psi^e_{mi}\phi_i{}^e(x, y), \qquad m = 1, 2, \tag{3.1}$$

$$\omega_m{}^e = \sum_{i=0}^{3} \omega^e_{mi}\phi_i{}^e(x, y), \qquad m = 1, 2, \tag{3.2}$$

where $\phi_0{}^e = (1 - X)(1 - Y)$, $\phi_1{}^e = X(1 - Y)$, $\phi_2{}^e = XY$, $\phi_3{}^e = (1 - X)Y$, $x = ih + hX$, $y = jh + hY$. The contribution of this element to the total functional is

$$J^e = (\psi^e_{1i}\omega^e_{1j} - \psi^e_{2i}\omega^e_{2j})\,\alpha^e_{ij} - \tfrac{1}{2}(\omega^e_{1i}\omega^e_{1j} - \omega^e_{2i}\omega^e_{2j})\,\gamma^e_{ij}$$
$$- (R/2)(\psi^e_{1i}\omega^e_{1j}\psi^e_{2k} - \psi^e_{2i}\omega^e_{2j}\psi^e_{1k})\,\beta^e_{ijk}, \tag{3.3}$$

where the double suffix notation is being used (no summation over e, 1, 2) and

$$\alpha^e_{ij} = \int_e \left(\frac{\partial\phi_i{}^e}{\partial x}\frac{\partial\phi_j{}^e}{\partial x} + \frac{\partial\phi_i{}^e}{\partial y}\frac{\partial\phi_j{}^e}{\partial y}\right) da, \qquad \gamma^e_{ij} = \int_e \phi_i{}^e\phi_j{}^e\, da$$

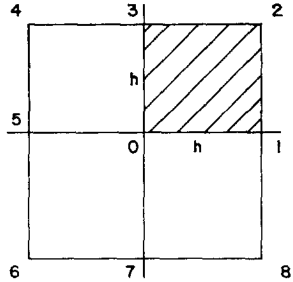

Fig. 2. Typical square element and node notation. Element e, origin 0 (ih, jh), local coordinates defined by $x = ih + Xh$, $y = jh + Yh$.

and

$$\beta^e_{ijk} = \int_e \left(\frac{\partial \phi_i{}^e}{\partial y} \frac{\partial \phi_j{}^e}{\partial x} - \frac{\partial \phi_j{}^e}{\partial y} \frac{\partial \phi_i{}^e}{\partial x} \right) \phi_k{}^e \, da. \tag{3.4}$$

The matrices α^e_{ij}, β^e_{ijk}, and γ^e_{ij} are

$$(\alpha^e_{ij}) = \frac{1}{6} \begin{pmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{pmatrix}, \qquad (\gamma^e_{ij}) = \frac{h^2}{36} \begin{pmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{pmatrix},$$

$$\begin{array}{c} \qquad\qquad k = 0 \qquad\qquad\qquad\qquad 1 \qquad\qquad\qquad\qquad 2 \qquad\qquad\qquad\qquad 3 \end{array}$$

$$(\beta^e_{ijk}) = \frac{1}{12} \left(\begin{array}{cccc|cccc|cccc|cccc} 0 & -2 & 0 & -2 & 0 & -2 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & -1 & 2 \\ 2 & 0 & -1 & -1 & 2 & 0 & -2 & 0 & 1 & 0 & -2 & 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 2 & 0 & -1 & 0 & 2 & 0 & -2 & 1 & 1 & 0 & -2 \\ -2 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & -1 & 2 & 0 & -2 & 0 & 2 & 0 \end{array} \right).$$

The condition that J is stationary implies that

$$\partial J/\partial \psi^e_{mi} = \partial J/\partial \omega^e_{mi} = 0, \qquad J = \sum_e J^e. \tag{3.5}$$

As $\psi_1 = \psi_2$ and $\omega_1 = \omega_2$ in the final solution these conditions can be used to simplify Eq. (3.5). With $\psi^e_{ni} = \Psi_i{}^e$, etc.,

$$\partial J^e/\partial \Psi_i{}^e = \alpha^e_{ij} \Omega_j{}^e - (R/2)(\beta^e_{ijk} - \beta^e_{kij}) \, \Omega_j{}^e \Psi_k{}^e, \tag{3.6}$$

$$\partial J^e/\partial \Omega_i{}^e = \alpha^e_{ij} \Psi_j{}^e - \gamma^e_{ij} \Omega_j{}^e - (R/2) \, \beta^e_{jik} \Psi_j{}^e \Psi_k{}^e. \tag{3.7}$$

The last term in Eq. (3.7) may be rewritten as

$$-(R/4)(\beta^e_{kij} + \beta^e_{jik}) \, \Psi_j{}^e \Psi_k{}^e = (R/4)(\beta^e_{ijk} - \beta^e_{kij}) \, \Psi_j{}^e \Psi_k{}^e,$$

using the symmetry of the product $\Psi_j{}^e \Psi_k{}^e$ and the antisymmetry of β_{ijk} with respect to its first two indices (the expression involving β^e_{ijk} is the same in (3.6) and (3.7)). At a typical node away from any boundaries and within a square mesh of size h these equations imply that

$$\begin{aligned} 0 = 8\Omega_0 - \sum_{i=1}^{8} \Omega_i \\ - \frac{R}{4} \{ & (\Psi_3 - \Psi_1)(\Omega_0 - \Omega_2) + (\Psi_5 - \Psi_3)(\Omega_0 - \Omega_4) + (\Psi_7 - \Psi_5)(\Omega_0 - \Omega_6) \\ & + (\Psi_1 - \Psi_7)(\Omega_0 - \Omega_8) + (\Psi_1 + \Psi_2 + \Psi_3)(\Omega_3 - \Omega_1) + (\Psi_3 + \Psi_4 + \Psi_5) \\ & \times (\Omega_5 - \Omega_3) + (\Psi_5 + \Psi_6 + \Psi_7)(\Omega_7 - \Omega_5) + (\Psi_7 + \Psi_8 + \Psi_1)(\Omega_1 - \Omega_7) \}, \end{aligned} \tag{3.8}$$

$$0 = 8\Psi_0 - \sum_{i=1}^{8} \Psi_i - \frac{h^2}{12} \left\{ 16\Omega_0 + 4 \sum_{i \,\mathrm{odd}} \Omega_i + \sum_{i \,\mathrm{even}} \Omega_i \right\}. \tag{3.9}$$

The nonlinear terms in (3.7) cancel when combined from all four elements surrounding node 0.

In each of (3.8) and (3.9) the first set of terms is the nonstandard 9-point second order approximation to the Laplacian. The term in brackets in (3.8) is identical to the second order Arakawa scheme for approximating the Jacobian of Ψ and Ω. The Jacobian is approximated by the sum $\frac{1}{3}(J_1 + J_2 + J_3)$ in the notation of Crow and Morton [5], an expression which is antisymmetric in Ψ and Ω and which in the absence of boundaries conserves the mean square vorticity and the mean square kinetic energy.

Near a boundary on which Ψ is constant Eq. (3.7) gives an approximation to the no-slip condition. If the boundary is parallel to the local y-axis and passes through node 0 and the fluid is in the region to the right of 0 the appropriate equation is

$$0 = 8\Psi_0 - 2(\Psi_1 + \Psi_2 + \Psi_8) - (\Psi_3 + \Psi_7) - (h^2/6)\{8\Omega_0 + 4\Omega_1 + 2(\Omega_3 + \Omega_7) + \Omega_2 + \Omega_8\}. \tag{3.10}$$

Again the nonlinear terms cancel. In the special case where both the stream function and vorticity do not depend on y Eq. (3.10) gives

$$0 = 3(\Psi_1 - \Psi_0) + h^2(\Omega_0 + \tfrac{1}{2}\Omega_1), \tag{3.11}$$

which is Woods' [6] method for wall vorticity, a commonly used second order accurate approximation.

4. Discussion

The variational principle for the stream function vorticity formulation of the Navier–Stokes equation has been used to derive some second order accurate finite element difference approximations. Simultaneously the no-slip conditions usually used at solid boundaries have been generalized. The method of this paper is currently being applied to a pressurized bearing problem.

The variational principles have been extended to time dependent flow problems and the resulting schemes are essentially identical to those recently discussed by Cullen [7, 8].

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